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Various Full Green Functions in QED

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Abstract We are interested in deriving various full Green functions through general Ward–Takahashi identities (WTIs) for quantized field theories. With the help of a postulate of gauge group parameter, the general local gauge transformation laws preserving the gauge-invariance of the generating functional itself of QED model have been established successfully. By using path-integral technique, the various WTIs with resulting anomaly terms are derived under the gauge transformations. The arising of Jacobian factor from the integration measure gives a viable possibility to express full Green function. As a consequence, the complete expressions of the full vector, the full axial-vector, the full tensor vertex functions and so on are presented respectively by solving the complete set of the WTIs in the momentum space without considering the constraint imposing any Ansatz. In addition, anomaly function also provides an effective means to judge the divergence of variant coupling currents on fields.

1 Introduction

It is a prominent feature of gauge field theories that gauge symmetry is inherited in quantized Lagrangian as the BRS symmetry [1–4]. WTIs connecting with the symmetry play an extremely important role in non-perturbative investigation of quantum field theory such as Dyson–Schwinger equation (DSE) approach to interpret the large amount of law-and intermediate energy phenomenology. Actually the current activity in DSE is centered around such a problem how to solve exactly the transverse part of the vertex function [5, 6].

It is important to note that property of Green function can be expressed through the WTI. For instance, the Ward–Takahashi identity $q_\mu \Gamma^\mu(k, p) = S^{-1}(k) - S^{-1}(p)$ imposing on the

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full vertex function can only specify the longitudinal part of the functions in QED [7]. Thus leaving the transverse component unconstrained becomes a challenging and open task in particle physics [8, 9], although there have been several attempts to construct the transverse part by Ansatz which satisfies some constraints not coming from the constraint imposed by the symmetry of the gauge invariance [10–13].

This situation may be attributed mainly to the fact that no established method can surface correctly to deal with the presentation of the transverse part of Green function (GF). Because there is little known about the precise information on the transverse part of Green function. Recently some progress on the problem has been made by using field operator approach grounded on the first principles of QED [9], in which the complete sets of Ward–Takahashi identities without anomaly are solved by computing the curl of the time-ordered products of three-point Green functions involving the vector, the axial-vector and the tensor current operators respectively. As such the transverse part of GF as well as the full vertices are exactly carried out in terms of fermion propagators without any Ansatz.

It is well-known that WT identity for axial-vector current is modified due to ABJ anomaly in the process of quantization, which may be in conflict with the symmetry apparent at the classical level [14, 15]. Therefore the presence of anomalous terms associated with the WTIs has been taken into account in a model of the quantized Dirac field with arbitrary internal degrees of freedom having arbitrary non-derivative coupling to external scalar 'pseudo scalar' vector and axial-vector fields by operator approach [16, 17]. The failure of the symmetry is expected at more fundamental level. This point is first realized by Fujikawa in the path-integral formulation of quantum field theory, for instance, the chiral-symmetry-breaking anomaly enters only in the integration measure used to define the path-integral over fermion fields [18–20]. In particular, a model corresponding to the axial-vector $U(1)$ gauge theory was considered in detail [18–21]. Apart from these results of anomaly, it is still desirable to know what types of currents can give rise to anomaly coming from the symmetry of quantized gauge theory itself (QED).

In this paper, we first investigate a new symmetry which stems from the gauge invariance of the generating functional under gauge transformations. This fact holds irrespective whether this re-naming field variables take the form of symmetry of action, or not. Consequently, a postulate for group parameter is introduced, which is a key to the derivation of various anomaly factors and of WTIs for variant fermionic currents. The group parameter as a scalar function can be expanded as a matrix series, so that the local gauge transformations naturally provide different types of interaction currents in the variation of the Lagrangian L_{eff} without loss of any rigorousness. Furthermore, a collection of the gauge transformations connected with the parameter postulate forms a Lie group.

As a firm basis for further study, we are devoted to deriving the correct anomaly terms arised from the quantum measure with respect to relevant transformations of field variables by generalizing Fujikawa's regulation method. By employing the proper regulator $f^d(-(\frac{\lambda_n}{M})^2)$ for the integration measure, trivial or non-trivial Jacobian for variant fermionic currents is evaluated respectively. Especially a new non-trivial Jacobian of high rank tensor current arises.

Based on the gauge invariance in a broad sense, the various WTIs with anomaly terms are derived explicitly, including a new class of WTIs for high rank tensor currents. We find the transverse competent of full Green function appears in the WT identities, which is a new finding different from the case of conventional Schwinger–Dyson equation [22]. As a consequence, these WTIs give the complete presentation of the full vector, the full axial-vector and the full tensor vertex functions.

In Sect. 2 the necessary QED formalism is briefly reviewed and the group parameter $\theta(x)$ of gauge transformation is expanded as a series combining tensor algebra function with Dirac matrix, which is fundamental to the following discussion. The subsequent Sect. 3 treats the topic of calculating anomaly terms, in which anomaly functions for variant fermionic currents have been expressed in detail. The following Sect. 4 shows how the new symmetry is employed to systematically present the various complete WTIs with anomaly terms under the local gauge transformations. Consequently, we derive the full vector, axial-vector and tensor vertex functions entirely coming from the sets of WTIs in Sect. 5. In the last section, some remarks on the comparison between our results and the previous investigations will be made.

2 Symmetry of Generating Function Itself

In studying the property of full Green function, it is extremely important that the symmetry transformation preserving the gauge invariance of generating functional itself leads to WTIs with anomaly. Therefore we now pay attention to gauge structure of the group parameter $\theta(x)$ as the basis of understanding the gauge invariance principle. Taken broadly, the changing variables of matter field can be transformed as [23]

$$\phi'_\alpha = \phi_\alpha + \delta\phi_\alpha, \tag{2.1}$$

$$\delta\phi_\alpha = \theta_{[\nu]}(x) H_\alpha^{[\nu]}(\phi_\beta, \phi_{\beta\lambda}) + \partial_\mu \theta_{[\nu]}(x) h_\alpha^{\mu[\nu]}(\phi_\beta, \phi_{\beta\lambda}) \tag{2.2}$$

where the matrices $H_\alpha^{[\nu]}, h_\alpha^{\mu[\nu]}$ are the functions composed of both Dirac matrix and field variables.

For infinitesimal gauge parameter $\theta(x)$, one finds that

$$i\theta(x)\phi_\alpha = \theta_{[\nu]}(x) H_\alpha^{[\nu]}(\phi_\beta, \phi_{\beta\lambda}) + \partial_\mu \theta_{[\nu]}(x) h_\alpha^{\mu[\nu]}(\phi_\beta, \phi_{\beta\lambda}). \tag{2.3}$$

As viewed from mathematics, a scalar function $\theta(x)$ can be expanded as a series with tensor functions in matrix form

$$\theta(x) = \theta_{[\nu]}(x) H^{[\nu]}, \tag{2.4}$$

where $\theta_{[\nu]}(x)$ are a set of arbitrary real tensor functions of x, the matrices $H^{[\nu]}$ denote respective coupling Dirac matrices $\Gamma^{[\nu]}$ (Dirac algebra is the algebra of 4×4 complex matrices). The dummy indices $[\nu]$ denote the index set. The above idea comes into being a physical interpretation. We naturally make an assumption relating to gauge transformation: The various couplings of interaction currents may occur in the process of mutual interaction of particles, the structure and the property of these currents can be expressed by the parameterized scalar function

$$\theta(x) = \theta_c \Gamma^c + \theta_\mu(x)\Gamma^\mu + \theta_{\mu\nu}(x)\Gamma^{\mu\nu} + \theta_{\mu\nu\rho}(x)\Gamma^{\mu\nu\rho} + \cdots. \tag{2.5}$$

Subsequently it is necessary to examine the property of the gauge group with the continuous parameter $\theta(x)$. By taking an operator $U(T(\theta(x)))$ as a representation for the symmetry transformations, the matter fields in physical Hilbert space transform as

$$\psi \to U(T(\theta(x)))\psi. \tag{2.6}$$

Herein we assume that the above symmetry transformations are the infinitesimal part of the Lie group. Thus the operators $U(T(\theta(x)))$ can be represented in at least a finite neighborhood of the identity by a power series

$$U(T(\theta(x))) = 1 + i\theta^a T^a + \frac{1}{2}\theta^a\theta^b T^a T^b + \cdots, \tag{2.7}$$

where $\theta^a(x)$ are a set of continuous parameters, T^a are generators relating to electric charge for the group and obey commutation relations $\lfloor T^a T^b \rfloor = 0$. Connecting with the parameterization equation (2.5), the group operators $U(\theta(x))$ change into

$$U(\theta(x)) = e^{i\theta^a(x)T^a} = e^{i(\theta_{[\nu]}H^{[\nu]})^a T^a}. \tag{2.8}$$

By using the general operator identity

$$e^{A+B} = e^A e^B e^{\frac{1}{2}[B,A]+\frac{1}{6}[[B,A],A]+\cdots} \tag{2.9}$$

and taking the form of the group multiplication law

$$U(\theta_1)U(\theta_2) = U(f(\theta_1,\theta_2)), \tag{2.10}$$

the law can be performed (for infinitesimal $\theta(x)$)

$$e^{i\theta_{[\mu]}(x)\Gamma^{[\mu]}} e^{i\theta_{[\nu]}(x)\Gamma^{[\nu]}} = e^{i\theta_{[\lambda]}(x)\Gamma^{[\lambda]}} e^{i\theta_{[\mu]}(x)\theta_{[\nu]}(x)[\Gamma^{[\mu]},\Gamma^{[\nu]}]+(\text{higher-order})} \tag{2.11}$$

where we have chosen to write the group parameter as $\theta_{[\nu]}\Gamma^{[\nu]}$. Owing to the parameter infinitesimal, the factors including the higher order $\theta_\mu\theta_\nu$ should be omitted. In virtue of the property of Dirac gamma matrices, a collection of the operators $U(\theta(x))$ meet the group properties such as closure, associativity, an identity element and an inverse.

Based on the above treatment, it is expected that the symmetry of the generating functional itself under the gauge transformations related to the postulate equation (2.5) will generate general class of WT identities.

3 Calculation of Anomaly Factor Coming from Integration Measure

According to Fujikawa's interpretation, it is argued that the appearance of the anomaly in WTI is a symptom of the impossibility of defining a suitably invariant functional integral measure with respect to the relevant transformations on fermionic field variables. The analysis based on the use of path integrals can provide access to a wider class of such anomaly objects.

We consider the theory with a mass fermion field $\psi(x)$ and a vector field $B^\mu(x)$, interacting through fermionic current $j^{[\nu]} = \bar{\psi}\Gamma^{[\nu]}\psi$. In sense of the functional integral, the underlying gauge theory (QED) possesses a new symmetry with the following transformation. By connecting with the above parametrization equations (2.5), the general localized infinitesimal gauge transformation rules for the fermion and gauge fields now take the form

$$\begin{aligned} \psi'(x) &= e^{-i\theta_{[\nu]}(x)\Gamma^{[\nu]}}\psi, \\ \bar{\psi}'(x) &= \bar{\psi}\gamma_0 e^{i\theta_{[\nu]}(x)\Gamma^{[\nu]+}}\gamma_0, \\ B'_\mu(x) &= B_\mu(x) - \partial_\mu\theta_{[\nu]}(x) \end{aligned} \tag{3.1}$$

where $\Gamma^{[\nu]}$ denote a set of an algebra terms being Dirac matrices (herein the repeated index $[\nu]$ is fixed). Furthermore, the variations of interaction part of Lagrangian L_{eff} would be presented in the form consistent with the gauge principle

$$\delta L_I = \bar{\psi}\gamma^\mu \delta B_{\mu[\nu]}(x)\Gamma^{[\nu]}\psi + \delta\bar{\psi}\gamma^\mu B_{\mu[\nu]}(x)\Gamma^{[\nu]}\psi + \bar{\psi}\gamma^\mu \delta B_{\mu[\nu]}(x)\Gamma^{[\nu]}\delta\psi. \tag{3.2}$$

Thus mutual interaction constructed in this way is valid for the symmetry transformation, which includes the usual case of localized gauge transformation.

Guided by the above considerations, the usual gauge-invariant QED Lagrangian density L_{eff} is modified to new form containing variant interaction currents consistent with (2.5),[1]

$$L_{\text{eff}} = \bar{\psi}i\gamma^\mu(\partial_\mu - iB_\mu(x)\Gamma^{[\nu']}\delta^{[\nu]}_{[\nu']})\psi - \bar{\psi}m\psi + L_G + L_{FG}, \tag{3.3}$$

$$L_G = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \qquad L_{FG} = -\frac{1}{2\xi}(\partial^\mu B_\mu)^2. \tag{3.4}$$

For convenience, the coupling constant $g = -e$ is suppressed and repeated indices are generally summed later on; where $\delta^{[\nu]}_{[\nu']}$ is the Kronecker delta, the upper index of which stands for the type of the corresponding transformation of fermionic current related to $\Gamma^{[\nu]}$, the lower index (dummy index) of which runs over all different kinds of interaction tensor currents in (2.5). In simple case, the fixed gauge term in (3.3) is dropped out by taking the special gauge ($\xi \to \infty$) [4].

We now turn to a calculation of anomaly in the transformation of the measure in sense of (3.1). The change in functional measure varies with the gauge transformations

$$d\mu \to d\mu' = \prod_n d\bar{c}'_n \prod_n dc'_n = (\det f_{nm})^{-1}(\det f'_{nm})^{-1}\prod_m d\bar{c}_m \prod_m dc_m \tag{3.5}$$

with

$$f_{nm} = \int d^4x\varphi^+_n(x)e^{-i\theta_{[\nu]}(x)\Gamma^{\{\nu\}}}\varphi_m(x). \tag{3.6}$$

The corresponding Jacobian factor becomes

$$(\det f'_{nm})^{-1} = e^{-i\int d^4x\gamma^0\Gamma^{[\nu]^+}(x)\gamma^0\theta_{[\nu]}(x)}, \tag{3.7}$$

where anomaly function $\Gamma^{[\nu]}(x)$ denotes the trace of Dirac matrix $\Gamma^{[\nu]}$ in the function space above

$$\Gamma^{[\nu]}(x) = \sum_n \varphi^+_n(x)\Gamma^{[\nu]}\varphi_n(x). \tag{3.8}$$

Based on the use of path integrals in Euclidean space, a general prescription to deal with a non-Hermitian operator can be applied to the modified operator by defining Hermitian operators [18–20, 24],

$$\begin{aligned} H_\psi &= \mathbb{D}(B)^+\mathbb{D}(B), \\ H_{\bar{\psi}} &= \mathbb{D}(B)\mathbb{D}(B)^+. \end{aligned} \tag{3.9}$$

[1] In the sense of the Minkowski space, metric tensor $g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ and the anticommutation relations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$.

As discussed in Refs. [18–20], regularization of the anomaly function is achieved by inserting the convergent factor $f^d(-(\frac{\lambda_n}{M})^2)$ and taking the limit as $M \to \infty$. To do this the above anomaly function can be written as the limit of a manifestly convergent integral

$$\Gamma^{[\nu]}(x) = \lim_{M\to\infty} \int \frac{d^4x}{(2\pi)^4} e^{-ikx} \Gamma^{[\nu]} f^d\left(\frac{-H_\psi}{M^2}\right) e^{ikx} \tag{3.10}$$

where power number d of the function $f(-(\frac{\lambda_n}{M})^2)$ takes an integer from 0 to 2, which is corresponding to tensor type of the fermionic currents related to matrix $\Gamma^{[\nu]}$.

In addition the transformation of the field $B_\mu(x)$ is a translation, so that its Jacobian is trivial.

Thus the Jacobian related to general transformations equations (3.1) can be put in the form

$$J = e^{i\int d^4x \Gamma^{[\nu]}(x)\theta_{[\nu]}(x)} e^{-i\int d^4x \bar{\Gamma}^{[\nu]}(x)\theta_{[\nu]}(x)} \tag{3.11}$$

with

$$\bar{\Gamma}^{[\nu]}(x) = \lim_{M\to\infty} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma^0 \Gamma^{[\nu]+} \gamma^0 f^d\left(\frac{-H_{\bar{\psi}}}{M^2}\right) e^{ikx}. \tag{3.12}$$

Based on the above expositions, it is straightforward to carry out the function $\Gamma(x)$. Now using the following operator identity consistent with (3.3), (3.9)

$$\begin{aligned} H_\psi = {} & (D_\mu)^2 + i\frac{1}{4}[\gamma_\mu\gamma_\nu]F_{\mu\nu}\Gamma^{[\nu]} + (B_\mu)^2\Gamma^{[\nu]^2} + \frac{1}{2}B_\mu(\gamma^\mu\gamma^\nu\Gamma^{[\nu]} + \gamma^\mu\Gamma^{[\nu]}\gamma^\nu)\partial_\nu \\ & + \frac{1}{2}B_\nu(\gamma^\nu\gamma^\mu\Gamma^{[\nu]} + \gamma^\nu\Gamma^{[\nu]}\gamma^\mu)\partial_\mu \\ & + \frac{1}{2}B_\mu B_\nu(\gamma^\mu\Gamma^{[\nu]}\gamma^\nu\Gamma^{[\nu]} + \gamma^\nu\Gamma^{[\nu]}\gamma^\mu\Gamma^{[\nu]}), \end{aligned} \tag{3.13}$$

the function $\Gamma^{[\nu]}(x)$ can be represented in terms of Taylor expansion by expanding it carefully in the form:

(1) For $d = 0$;

$$f^0\left(\frac{-H_\psi}{M^2}\right) = 1, \tag{3.14}$$

this shows that the functional measure needs no regularization.

(2) For $d = 1$; the case is corresponding to first-rank tensor function $\theta_{[\nu]}(x)$,

$$\Gamma^{[\nu]}(x) = \frac{-i}{2^9\pi^2}\operatorname{tr}\{\Gamma^{[\nu]}[\gamma_\mu\gamma_\nu]\Gamma^{[\nu]}[\gamma_\rho\gamma_\sigma]\Gamma^{[\nu]}F_{\mu\nu}F_{\rho\sigma}\} \tag{3.15}$$

where the higher order terms have been dropped out in the expansion.

(3) For $d = 2$; corresponding to high rank tensor function (rank ≥ 2)

$$\Gamma^{[\nu]}(x) = \frac{-i}{4^5\pi^2}\operatorname{tr}\{\Gamma^{[\nu]}[\gamma_\mu\gamma_\nu]\Gamma^{[\nu]}[\gamma_\rho\gamma_\sigma]\Gamma^{[\nu]}F_{\mu\nu}F_{\rho\sigma}\}. \tag{3.16}$$

It comes to our notice that if the proper regulator $f(-(\frac{\lambda_n}{M})^2)$ is chosen, (3.15) would be transmuted into the same form as (3.16). These processes of the evaluation suggest that the form of the employed regulator functions may be not of unique, the resulting anomaly depends on the choice of the operator $\mathbb{D}$.

Written in the above fashion, the anomaly functions $\Gamma^{[\nu]}(x)$ depending on matrix $\Gamma^{[\nu]}$ in (2.5) and its corresponding Jacobian are evaluated below:

$$\text{(i)}\quad \Gamma^{[1]}=1,\qquad \Gamma^{[1]}(x)=\frac{-i}{2^9\pi^2}\,\mathrm{tr}(F_{\mu\nu}F_{\lambda\rho})(g_{\mu\rho}g_{\nu\lambda}-g_{\mu\lambda}g_{\nu\rho}),\qquad J_{[1]}=1. \tag{3.17}$$

$$\text{(ii)}\quad \Gamma^{[5]}=\gamma^{[5]},\qquad \Gamma^{[5]}(x)=\frac{i}{16\pi^2}\,\mathrm{tr}(F_{\mu\nu}\tilde F_{\rho\sigma})\quad\text{with } \tilde F_{\rho\sigma}(x)=-\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu},$$
$$J_{[5]}=\exp\left[2i\int d^4x\theta_5(x)\left(\frac{i}{16\pi^2}\,\mathrm{tr}(F_{\mu\nu}\tilde F_{\rho\sigma})\right)\right]. \tag{3.18}$$

$$\text{(iii)}\quad \Gamma^{[\nu]}=\gamma^{\mu},\qquad \Gamma^{[\mu]}(x)=0,\qquad J_{[\mu]}=1. \tag{3.19}$$

$$\text{(iv)}\quad \Gamma^{[\mu5]}=\gamma^{\mu}\gamma_5,\qquad \Gamma^{[\mu5]}(x)=0,\qquad J_{[\mu5]}=1. \tag{3.20}$$

$$\text{(v)}\quad \Gamma^{[\lambda\mu\nu5]}=\varepsilon^{\lambda\mu\nu\rho}\gamma_\rho\gamma_5,\qquad \Gamma^{[\lambda\mu\nu]}(x)=0,\qquad J_{[\lambda\mu\nu]}=1. \tag{3.21}$$

$$\text{(vi)}\quad \Gamma^{[\mu\nu]}=\sigma^{\mu\nu},$$

$$\begin{aligned}
\Gamma^{[\mu\nu]}(x)={}&\frac{i\mathrm{Tr}[F^{\mu\nu}F^{\rho\sigma}]}{64\pi^2}\\
&\times\{-g_{\mu\alpha}g_{\nu\beta}(-ig_{\alpha_1\rho}(g_{\beta_1\alpha_2}g_{\sigma\beta_2}-g_{\beta_1\beta_2}g_{\sigma\alpha_2})+ig_{\alpha_1\sigma}(g_{\beta_1\alpha_2}g_{\sigma\beta_2}-g_{\beta_1\beta_2}g_{\sigma\alpha_2})\\
&+ig_{\beta_1\rho}(g_{\beta_1\alpha_1}g_{\sigma\beta_2}-g_{\beta_1\sigma}g_{\rho\alpha_1})-ig_{\beta_1\sigma}(g_{\alpha_1\alpha_2}g_{\rho\beta_2}-g_{\alpha_1\beta_2}g_{\rho\alpha_2}))\\
&-g_{\mu\beta}g_{\nu\alpha}(-ig_{\alpha_1\rho}(g_{\beta_1\alpha_2}g_{\sigma\beta_2}-g_{\beta_1\beta_2}g_{\sigma\alpha_2})+ig_{\alpha_1\sigma}(g_{\beta_1\alpha_2}g_{\sigma\beta_2}-g_{\beta_1\beta_2}g_{\sigma\alpha_2})\\
&+ig_{\beta_1\rho}(g_{\beta_1\alpha_1}g_{\sigma\beta_2}-g_{\beta_1\sigma}g_{\rho\alpha_1})-ig_{\beta_1\sigma}(g_{\alpha_1\alpha_2}g_{\rho\beta_2}-g_{\alpha_1\beta_2}g_{\rho\alpha_2}))\\
&-i\varepsilon_{\mu\nu\alpha\beta}(-g_{\alpha_1\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2}+g_{\alpha_1\sigma}\varepsilon_{\mu\nu\alpha\beta}+g_{\beta_1\rho}\varepsilon_{\alpha_2\beta_2\alpha_1\sigma}-g_{\beta_1\sigma}\varepsilon_{\alpha_1\rho\alpha_2\beta_2})\\
&+ig_{\mu\alpha}[-g_{\alpha_1\nu}g_{\beta_1\beta}(g_{\rho\alpha_2}g_{\sigma\beta_2}-g_{\rho\beta_2}g_{\sigma\alpha_2})+g_{\alpha_1\beta}g_{\beta_1\nu}(g_{\rho\alpha_2}g_{\sigma\beta_2}-g_{\rho\beta_2}g_{\sigma\alpha_2})\\
&+g_{\alpha_1\nu}g_{\beta_1\rho}(g_{\beta\alpha_2}g_{\sigma\beta_2}-g_{\beta\beta_2}g_{\sigma\alpha_2})-g_{\alpha_1\beta}g_{\beta_1\rho}(g_{\nu\alpha_2}g_{\sigma\beta_2}-g_{\nu\beta_2}g_{\sigma\alpha_2})\\
&-g_{\beta\nu}g_{\alpha_1\rho}(g_{\beta_1\alpha_2}g_{\sigma\beta_2}-g_{\beta\beta_2}g_{\sigma\alpha_2})+g_{\alpha_1\rho}g_{\beta_1\beta}(g_{\nu\alpha_2}g_{\sigma\beta_2}-g_{\nu\beta_2}g_{\sigma\alpha_2})]\\
&-ig_{\mu\beta}[-g_{\alpha_1\nu}g_{\beta_1\alpha}(g_{\rho\alpha_2}g_{\sigma\beta_2}-g_{\rho\beta_2}g_{\sigma\alpha_2})+g_{\alpha_1\alpha}g_{\beta_1\nu}(g_{\rho\alpha_2}g_{\sigma\beta_2}-g_{\rho\beta_2}g_{\sigma\alpha_2})\\
&+g_{\alpha_1\nu}g_{\beta_1\rho}(g_{\alpha\alpha_2}g_{\sigma\beta_2}-g_{\alpha\beta_2}g_{\sigma\alpha_2})-g_{\alpha_1\alpha}g_{\beta_1\rho}(g_{\nu\alpha_2}g_{\sigma\beta_2}-g_{\nu\beta_2}g_{\sigma\alpha_2})\\
&-g_{\beta_1\nu}g_{\alpha_1\rho}(g_{\alpha\alpha_2}g_{\sigma\beta_2}-g_{\beta\beta_2}g_{\sigma\alpha_2})+g_{\alpha_1\rho}g_{\beta_1\alpha}(g_{\nu\alpha_2}g_{\sigma\beta_2}-g_{\nu\beta_2}g_{\sigma\alpha_2})]\\
&-ig_{\nu\alpha}[-g_{\alpha_1\mu}g_{\beta_1\beta}(g_{\rho\alpha_2}g_{\sigma\beta_2}-g_{\rho\beta_2}g_{\sigma\alpha_2})+g_{\alpha_1\beta}g_{\beta_1\mu}(g_{\rho\alpha_2}g_{\sigma\beta_2}-g_{\rho\beta_2}g_{\sigma\alpha_2})\\
&+g_{\alpha_1\mu}g_{\beta_1\rho}(g_{\beta\alpha_2}g_{\sigma\beta_2}-g_{\beta\beta_2}g_{\sigma\alpha_2})-g_{\alpha_1\beta}g_{\beta_1\rho}(g_{\mu\alpha_2}g_{\sigma\beta_2}-g_{\mu\beta_2}g_{\sigma\alpha_2})\\
&-g_{\beta\mu}g_{\alpha_1\rho}(g_{\beta_1\alpha_2}g_{\sigma\beta_2}-g_{\beta\beta_2}g_{\sigma\alpha_2})+g_{\alpha_1\rho}g_{\beta_1\beta}(g_{\mu\alpha_2}g_{\sigma\beta_2}-g_{\mu\beta_2}g_{\sigma\alpha_2})]\\
&+ig_{\nu\beta}[-g_{\alpha_1\mu}g_{\beta_1\alpha}(g_{\rho\alpha_2}g_{\sigma\beta_2}-g_{\rho\beta_2}g_{\sigma\alpha_2})+g_{\alpha_1\alpha}g_{\beta_1\mu}(g_{\rho\alpha_2}g_{\sigma\beta_2}-g_{\rho\beta_2}g_{\sigma\alpha_2})\\
&+g_{\alpha_1\mu}g_{\beta_1\rho}(g_{\alpha\alpha_2}g_{\sigma\beta_2}-g_{\alpha\beta_2}g_{\sigma\alpha_2})-g_{\alpha_1\alpha}g_{\beta_1\rho}(g_{\mu\alpha_2}g_{\sigma\beta_2}-g_{\mu\beta_2}g_{\sigma\alpha_2})\\
&-g_{\alpha\mu}g_{\alpha_1\rho}(g_{\beta_1\alpha_2}g_{\sigma\beta_2}-g_{\beta\beta_2}g_{\sigma\alpha_2})+g_{\alpha_1\rho}g_{\beta_1\alpha}(g_{\mu\alpha_2}g_{\sigma\beta_2}-g_{\mu\beta_2}g_{\sigma\alpha_2})]\},\\
J_{[\mu\nu]}={}&0,
\end{aligned} \tag{3.22}$$

(vii) $\Gamma^{[\mu\nu 5]} = \sigma^{\mu\nu}\gamma_5$,

$$
\begin{aligned}
\Gamma^{[\mu\nu 5]}(x) = \frac{\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})}{-i64\pi^2}
&\times [g_{\mu\alpha}g_{\nu\beta}(-g_{\alpha_1\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} + g_{\alpha_1\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} + g_{\beta_1\rho}\varepsilon_{\alpha_2\beta_2\alpha_1\sigma} - g_{\beta_1\sigma}\varepsilon_{\alpha_1\rho\alpha_2\beta_2}) \\
&- g_{\mu\beta}g_{\nu\alpha}(-g_{\alpha_1\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} + g_{\alpha_1\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} + g_{\beta_1\rho}\varepsilon_{\alpha_2\beta_2\alpha_1\sigma} - g_{\beta_1\sigma}\varepsilon_{\alpha_1\rho\alpha_2\beta_2}) \\
&+ \varepsilon_{\mu\nu\alpha\beta}(-g_{\alpha_1\rho}(g_{\beta_1\alpha_2}g_{\sigma\beta_2} - g_{\beta_1\beta_2}g_{\sigma\alpha_2}) + g_{\alpha_1\rho}(g_{\beta_1\alpha_2}g_{\rho\beta_2} - g_{\beta_1\beta_2}g_{\rho\alpha_2}) \\
&+ g_{\beta_1\rho}(g_{\alpha_1\alpha_2}g_{\sigma\beta_2} - g_{\alpha_1\beta_2}g_{\sigma\alpha_{21}}) - g_{\beta_1\sigma}(g_{\alpha_1\alpha_2}g_{\rho\beta_2} - g_{\alpha_1\beta_2}g_{\rho\alpha_2})) \\
&+ g_{\mu\alpha}[-g_{\alpha_1\nu}g_{\beta_1\beta}\varepsilon_{\rho\sigma\alpha_2\beta_2} + g_{\alpha_1\beta}g_{\beta_1\nu}\varepsilon_{\rho\sigma\alpha_2\beta_2} - g_{\rho\alpha_2}g_{\sigma\beta_2}\varepsilon_{\alpha_1\beta_1\nu\beta} \\
&+ g_{\rho\beta_2}g_{\sigma\alpha_2}\varepsilon_{\alpha_1\beta_1\nu\beta} + g_{\alpha_1\nu}g_{\beta_1\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} - g_{\alpha_1\nu}g_{\beta_1\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} - g_{\alpha_1\nu}g_{\beta\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} \\
&+ g_{\alpha_1\nu}g_{\beta\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} + g_{\alpha_1\beta}g_{\beta_1\rho}\varepsilon_{\nu\sigma\alpha_2\beta_2} - g_{\alpha_1\nu}g_{\beta_1\sigma}\varepsilon_{\nu\rho\alpha_2\beta_2} - g_{\alpha_1\beta}g_{\nu\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} \\
&+ g_{\alpha_1\beta}g_{\nu\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} - g_{\beta_1\nu}g_{\alpha_1\rho}\varepsilon_{\beta\sigma\alpha_2\beta_2} + g_{\beta_1\nu}g_{\alpha_1\sigma}\varepsilon_{\beta\rho\alpha_2\beta_2} + g_{\beta_1\nu}g_{\beta\rho}\varepsilon_{\alpha_1\sigma\alpha_2\beta} \\
&- g_{\beta_1\nu}g_{\beta\rho}\varepsilon_{\alpha_1\rho\alpha_2\beta_2} + g_{\beta_1\beta}g_{\alpha_1\rho}\varepsilon_{\nu\sigma\alpha_2\beta_2} - g_{\beta_1\beta}g_{\alpha_1\sigma}\varepsilon_{\nu\rho\alpha_2\beta_2} - g_{\beta_1\beta}g_{\nu\rho}\varepsilon_{\alpha_1\sigma\alpha_2\beta} \\
&- g_{\beta_1\nu}g_{\sigma\rho}\varepsilon_{\alpha_1\rho\alpha_2\beta_2}] \\
&- g_{\mu\beta}[-g_{\alpha_1\nu}g_{\beta_1\alpha}\varepsilon_{\rho\sigma\alpha_2\beta_2} + g_{\alpha_1\alpha}g_{\beta_1\nu}\varepsilon_{\rho\sigma\alpha_2\beta_2} - g_{\rho\alpha_2}g_{\sigma\beta_2}\varepsilon_{\alpha_1\beta_1\nu\alpha} \\
&+ g_{\rho\beta_2}g_{\sigma\alpha_2}\varepsilon_{\alpha_1\beta_1\nu\alpha} + g_{\alpha_1\nu}g_{\beta_1\rho}\varepsilon_{\alpha\sigma\alpha_2\beta_2} - g_{\alpha_1\nu}g_{\beta_1\sigma}\varepsilon_{\alpha\rho\alpha_2\beta_2} - g_{\alpha_1\nu}g_{\alpha\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} \\
&+ g_{\alpha_1\nu}g_{\alpha\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} + g_{\alpha_1\alpha}g_{\beta_1\rho}\varepsilon_{\nu\sigma\alpha_2\beta_2} - g_{\alpha_1\nu}g_{\beta_1\sigma}\varepsilon_{\nu\rho\alpha_2\beta_2} - g_{\alpha_1\beta}g_{\nu\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} \\
&+ g_{\alpha_1\beta}g_{\nu\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} - g_{\beta_1\nu}g_{\alpha_1\rho}\varepsilon_{\alpha\sigma\alpha_2\beta_2} + g_{\beta_1\nu}g_{\alpha_1\sigma}\varepsilon_{\alpha\rho\alpha_2\beta_2} + g_{\beta_1\nu}g_{\alpha\rho}\varepsilon_{\alpha_1\sigma\alpha_2\beta} \\
&- g_{\beta_1\nu}g_{\alpha\rho}\varepsilon_{\alpha_1\rho\alpha_2\beta_2} + g_{\beta_1\alpha}g_{\alpha_1\rho}\varepsilon_{\nu\sigma\alpha_2\beta_2} - g_{\beta_1\alpha}g_{\alpha_1\sigma}\varepsilon_{\nu\rho\alpha_2\beta_2} - g_{\beta_1\alpha}g_{\nu\rho}\varepsilon_{\alpha_1\sigma\alpha_2\beta} \\
&- g_{\beta_1\alpha}g_{\sigma\rho}\varepsilon_{\alpha_1\rho\alpha_2\beta_2}] \\
&- g_{\nu\alpha}[-g_{\alpha_1\mu}g_{\beta_1\beta}\varepsilon_{\rho\sigma\alpha_2\beta_2} + g_{\alpha_1\beta}g_{\beta_1\mu}\varepsilon_{\rho\sigma\alpha_2\beta_2} - g_{\rho\alpha_2}g_{\sigma\beta_2}\varepsilon_{\alpha_1\beta_1\mu\beta} \\
&+ g_{\rho\beta_2}g_{\sigma\alpha_2}\varepsilon_{\alpha_1\beta_1\mu\beta} + g_{\alpha_1\mu}g_{\beta_1\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} - g_{\alpha_1\mu}g_{\beta_1\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} - g_{\alpha_1\mu}g_{\beta\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} \\
&+ g_{\alpha_1\mu}g_{\beta\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} + g_{\alpha_1\beta}g_{\beta_1\rho}\varepsilon_{\mu\sigma\alpha_2\beta_2} - g_{\alpha_1\nu}g_{\beta_1\sigma}\varepsilon_{\mu\rho\alpha_2\beta_2} - g_{\alpha_1\beta}g_{\mu\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} \\
&+ g_{\alpha_1\beta}g_{\mu\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} - g_{\beta_1\mu}g_{\alpha_1\rho}\varepsilon_{\beta\sigma\alpha_2\beta_2} + g_{\beta_1\mu}g_{\alpha_1\sigma}\varepsilon_{\beta\rho\alpha_2\beta_2} + g_{\beta_1\mu}g_{\beta\rho}\varepsilon_{\alpha_1\sigma\alpha_2\beta} \\
&- g_{\beta_1\mu}g_{\beta\rho}\varepsilon_{\alpha_1\rho\alpha_2\beta_2} + g_{\beta_1\beta}g_{\alpha_1\rho}\varepsilon_{\mu\sigma\alpha_2\beta_2} - g_{\beta_1\beta}g_{\alpha_1\sigma}\varepsilon_{\mu\rho\alpha_2\beta_2} - g_{\beta_1\beta}g_{\mu\rho}\varepsilon_{\alpha_1\sigma\alpha_2\beta} \\
&- g_{\beta_1\mu}g_{\sigma\rho}\varepsilon_{\alpha_1\rho\alpha_2\beta_2}] \\
&+ g_{\nu\beta}[-g_{\alpha_1\mu}g_{\beta_1\alpha}\varepsilon_{\rho\sigma\alpha_2\beta_2} + g_{\alpha_1\alpha}g_{\beta_1\mu}\varepsilon_{\rho\sigma\alpha_2\beta_2} - g_{\rho\alpha_2}g_{\sigma\beta_2}\varepsilon_{\alpha_1\beta_1\mu\alpha} \\
&+ g_{\rho\beta_2}g_{\sigma\alpha_2}\varepsilon_{\alpha_1\beta_1\mu\alpha} + g_{\alpha_1\mu}g_{\beta_1\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} - g_{\alpha_1\mu}g_{\beta_1\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} - g_{\alpha_1\mu}g_{\alpha\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} \\
&+ g_{\alpha_1\mu}g_{\alpha\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} + g_{\alpha_1\alpha}g_{\beta_1\rho}\varepsilon_{\mu\sigma\alpha_2\beta_2} - g_{\alpha_1\mu}g_{\beta_1\sigma}\varepsilon_{\mu\rho\alpha_2\beta_2} - g_{\alpha_1\alpha}g_{\mu\rho}\varepsilon_{\beta_1\sigma\alpha_2\beta_2} \\
&+ g_{\alpha_1\alpha}g_{\mu\sigma}\varepsilon_{\beta_1\rho\alpha_2\beta_2} - g_{\beta_1\mu}g_{\alpha_1\rho}\varepsilon_{\alpha\sigma\alpha_2\beta_2} + g_{\beta_1\mu}g_{\alpha_1\sigma}\varepsilon_{\alpha\rho\alpha_2\beta_2} + g_{\beta_1\mu}g_{\alpha\rho}\varepsilon_{\alpha_1\sigma\alpha_2\beta} \\
&- g_{\beta_1\mu}g_{\alpha\rho}\varepsilon_{\alpha_1\rho\alpha_2\beta_2} + g_{\beta_1\alpha}g_{\alpha_1\rho}\varepsilon_{\mu\sigma\alpha_2\beta_2} - g_{\beta_1\alpha}g_{\alpha_1\sigma}\varepsilon_{\mu\rho\alpha_2\beta_2} - g_{\beta_1\alpha}g_{\mu\rho}\varepsilon_{\alpha_1\sigma\alpha_2\beta} \\
&- g_{\beta_1\alpha}g_{\sigma\mu}\varepsilon_{\alpha_1\rho\alpha_2\beta_2}]],
\end{aligned}
$$

$$J_{[\mu\nu 5]} = e^{2i\int d^4x\theta_{\mu\nu}(x)(\Gamma^{[\mu\nu 5]}(x))}. \tag{3.23}$$

In the above calculation, Dirac matrix's properties have been used sufficiently. It is easy to see that the Jacobian arised from ψ and that from $\bar{\psi}$ cancel to each other in the cases of $\Gamma^{[1]} = 1$ and $\Gamma^{[\mu\nu]} = \sigma^{\mu\nu}$. In particular, a non-trivial Jacobian equation (3.2) of high rank pseudo-tensor current arises, its effect is not clear yet.

In fact, the general expressions of $\Gamma^{[\nu]}(x)$ also show that path-integral method is appropriate regularization way to judge the divergence of variant fermion currents related to the gauge transformations of field variables, which is quite useful in the derivation of WT identities.

4 Derivation of Ward–Takahashi Identities

We are now in a position to derive general Ward–Takahashi identities associated with the variations of field variables (2.5) in a broad sense. The generating functional can serve as a starting point for the non-perturbative treatment of the theory. We are thus led to consider the theory described by the generating functional with external sources J^μ, $\bar{\eta}$ and η:

$$Z[\bar{\eta}\eta J^\mu] = \int D[\bar{\psi}\psi B_\mu] \exp\left\{ i \int d^4x (L_{\text{eff}} + J^\mu B_\mu + \bar{\eta}\psi + \bar{\psi}\eta) \right\}. \tag{4.1}$$

In virtue of the symmetry, the variation of the generating functional with respect to $\theta_{[\nu]}(x)$ in (2.5) give a generation equation

$$\begin{aligned} &\int D[\bar{\psi}\psi B_\mu] \exp\left\{ iI + i \int d^4x (J^\mu B_\mu + \bar{\eta}\psi + \bar{\psi}\eta) \right\} \frac{\delta}{\delta\theta_{[\nu]}(x)} \\ &\quad \times \left(J + i\delta I_{\text{eff}} + i \int d^4x (J^\mu \delta B_\mu + \bar{\eta}\delta\psi + \delta\bar{\psi}\eta) \right) \Bigg|_{\theta_{[\nu]}(x)=0} = 0 \end{aligned} \tag{4.2}$$

where the variation of the generating functional is parameterized by $\theta_{[\nu]}(x)$ (all of $\theta_{[\nu]}(x)$ are regarded as formally independent functions to each other). The symbol J denotes a Jacobian factor of the transformation measure, the variation of the action I_{eff} is derived from the variations of field variables.

Usually, the generation equation (4.3) for Ward identity can be regarded as a certain combination of the Schwinger–Dyson equations; however, it possesses a new feature in sense of the transformation equation (3.1). These identities will give a viable possibility to present the complete vertex function.

Subsequently, with the choice of (2.17)

$$\begin{aligned} \theta(x) = {} & \theta_c I + \theta(x) I + \theta_\mu(x) i\gamma^\mu + \theta_5(x)\gamma^5 + \theta_{\mu 5}(x)\gamma^\mu\gamma^5 \\ & + \theta_{\mu\nu 5}(x) i\sigma^{\mu\nu}\gamma^5 + \theta_{\mu\nu}(x) i\sigma^{\mu\nu} + \theta_{\mu\nu\kappa 5}(x)\varepsilon^{\mu\nu\kappa\rho}\gamma_\rho\gamma^5 + \cdots. \end{aligned} \tag{4.3}$$

The following WT identities are derived correctly below under the general infinitesimal transformations of variables (3.1):

(i) Differentiation by tensor function

$$\theta_{[\nu]}(x) = \theta_c. \tag{4.4}$$

The functional differentiation with respect to the constant parameter θ_c in sense of (4.2) implies just the Abelian global gauge transformation. The WT identity corresponding to the transformation current $j_c(x) = \bar{\psi} I \psi$ can be written as

$$\int D[\bar{\psi}\psi B_\mu] i \int d^4x[-\delta(x-x_2)\bar{\psi}(x_1)\psi(x) + \delta(x-x_1)\bar{\psi}(x)\psi(x_2)]e^{i\sigma} = 0, \tag{4.5}$$

where the corresponding anomaly function $\Gamma^{[1]}(x)$ has been taken in account.

The identity can be written in more clear form in the momentum space

$$S(p_1) = S(p_2), \tag{4.6}$$

where $S(p_1)$ is the complete propagator of fermion. This means that there have no mutual interactions between fermions and gauge field.

$$\text{(ii)} \quad \theta_{[\nu]}(x) = \theta(x). \tag{4.7}$$

The variation of (4.3) with respect to the parameter $\theta(x)$ is equivalent to the following local gauge transformations on the fields

$$\begin{aligned} \delta\psi &= -i\theta_c(x) I \psi(x), \\ \delta\bar{\psi} &= +i\bar{\psi}\theta_c(x) I, \\ \delta B_\mu(x) &= -\partial_\mu \theta(x). \end{aligned} \tag{4.8}$$

We have the familiar WT identity for fermionic vector current for $j^\mu(x) = \bar{\psi}\gamma^\mu\psi$

$$q^\mu \Gamma_\mu(p_1 p_2 q) = S^{-1}(p_1) - S(p_2)^{-1}, \tag{4.9}$$

where $q = p_1 - p_2$, $\Gamma_{V\mu}$ is the vector vertex function.

We notice that anomalous term associated with this identity vanishes because all the divergent currents in $\delta L'_{\text{eff}}$ cancel against each other.

$$\text{(iii)} \quad \theta_{[\nu]}(x) = \theta_\mu(x). \tag{4.10}$$

In the case, there is a anomalous contribution coming from divergence of Noether currents, not from integral measure

$$\begin{aligned} A(x) = {} & \frac{-i}{2^8\pi^2}\partial_\mu \operatorname{tr}(F_{\mu\nu}F_{\lambda\rho})(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) \\ & - \frac{B(x)}{2^8\pi^2}\operatorname{tr}(F_{\mu\nu}F_{\lambda\rho})(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}). \end{aligned} \tag{4.11}$$

The corresponding WTI reads off in coordinate space for

$$\begin{aligned} j^{\mu\nu}(x) = {} & \bar{\psi}\gamma^\mu\gamma^\nu\psi 2i\partial_\lambda \langle 0|T(\bar{\psi}g^{\lambda\mu}\psi\bar{\psi}(x_1)\psi(x_2) - i\bar{\psi}\sigma^{\lambda\mu}\psi\bar{\psi}(x_1)\psi(x_2))|0\rangle \\ & - 2m\langle 0|T(\bar{\psi}\gamma^\mu\psi\bar{\psi}(x)\psi(x_2))|0\rangle - i\delta(x-x_2)\langle 0|T(\bar{\psi}(x_1)\psi(x))|0\rangle \\ & - i\delta(x-x_1)\langle 0|T(\bar{\psi}(x)\psi(x_2))|0\rangle + \langle 0|T(A(x)\bar{\psi}(x)\psi(x_2))|0\rangle = 0. \end{aligned} \tag{4.12}$$

This identity can be rewritten in momentum space

$$2iq_\lambda \Gamma^{\lambda\mu}(p_1p_2) = q^\mu\Gamma_s + S^{-1}(p_2)\gamma^\mu + S(p_1)^{-1}\gamma^\mu + 2m\Gamma_V^\mu(p_1p_2) + A(q), \tag{4.13}$$

where Γ_s, Γ_V are scalar and vector vertex functions respectively. Actually, the above equation is the longitudinal WTI of the tensor vertex function.

$$\text{(iv)} \quad \theta_{[\nu]}(x) = \theta_5(x). \tag{4.14}$$

Axial current and axial-vector current are defined by

$$\begin{aligned} j_5^{\mu}(x) &= \bar{\psi}\gamma^{\mu}\gamma_5\psi, \\ j_5(x) &= \bar{\psi}i\gamma_5\psi \end{aligned} \tag{4.15}$$

where the Hermitian matrix γ_5 is defined by

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{4.16}$$

The corresponding WTI reads off in coordinate space

$$\begin{aligned} &\partial_\lambda\langle 0|T(\bar{\psi}i\gamma^{\lambda}\gamma_5\psi\bar{\psi}(x_1)\psi(x_2))|0\rangle + \langle 0|T(A(x)\bar{\psi}(x_1)\psi(x_2))|0\rangle \\ &\quad - 2m\langle 0|T(\bar{\psi}i\gamma_5\psi\bar{\psi}(x_1)\psi(x_2))|0\rangle + \gamma_5\delta(x-x_2)\langle 0|T(\bar{\psi}(x_1)\psi(x))|0\rangle \\ &\quad + \delta(x-x_1)\gamma_5\langle 0|T(\bar{\psi}(x)\psi(x_2))|0\rangle = 0. \end{aligned} \tag{4.17}$$

In the momentum space, it changes to the following form

$$q_\lambda\Gamma_5^{\lambda}(p_1p_2q) = -i2m\Gamma_5(p_1p_2) + \gamma_5S^{-1}(p_2) + \gamma_5S^{-1}(p_1) - J_{[5]}(q) \tag{4.18}$$

where Γ_5 is the pseudo-scalar function. The $J_{[5]}(q)$ is the Fourier expression of $J_{[5]}(x)$. This is non-other than the usual axial-vector WTI in the literature [1–3, 9].

Indeed, this equation verified the usual conclusion that the axial current conservation contain an anomaly if gauge field involves γ_5 couplings.

(v) In particular, there is a special case of no coupling interaction between fermions and gauge fields. By making the change of variables

$$\begin{aligned} \delta\psi &= i\theta(x)\gamma_5\psi(x), \\ \delta\bar{\psi} &= i\bar{\psi}\theta(x)\gamma_5, \\ \delta B_\lambda(x) &= 0. \end{aligned} \tag{4.19}$$

Similarly, WT identity reads off in coordinate space for

$$\begin{aligned} j_5^{\mu}(x) = \bar{\psi}i\gamma^{\mu}\gamma_5\psi&\langle 0|T(J_{[5]}(x)\bar{\psi}(x_1)\psi(x_2))|0\rangle \\ &+ \gamma_5\delta(x-x_2)\langle 0|T(\bar{\psi}(x_1)\psi(x))|0\rangle \\ &+ \delta(x-x_1)\gamma_5\langle 0|T(\bar{\psi}(x)\psi(x_2))|0\rangle = 0 \end{aligned} \tag{4.20}$$

where the factor $J_{[5]}(x)$ is just the anomaly factor equation (3.18) associated with the current j_5^{μ}.

By virtue of the Dirac equations with m, this is taken into account carefully because of the divergent currents

$$\begin{aligned} (i\gamma^{\mu}\overrightarrow{D}_\mu - m)\psi(x) &= 0, \\ \bar{\psi}(x)(i\gamma^{\mu}\overleftarrow{D}_\mu + m) &= 0, \end{aligned} \tag{4.21}$$

the anomalous WTI can be written as

$$0 = J_{[5]}(q) + i\gamma_5 S^{-1}(p_2) + i\gamma_5 S^{-1}(p_1). \tag{4.22}$$

The physical relevance of this equation is not apparent at the present time. The resulting equation give a possible explanation that the effect of the anomaly composed of vector field $B^\mu(x)$ is to preserve a connection between fermion fields and vector field.

$$\text{(vi)} \quad \theta_{[\nu]}(x) = \theta_{5\mu}(x). \tag{4.23}$$

Performing the same procedure for the $j_5^{\lambda\mu}(x) = \bar{\psi}\gamma^\lambda\gamma^\mu\gamma_5\psi$, WT identity is

$$\begin{aligned}&\partial_\lambda\langle 0|T(\bar{\psi}\gamma^\lambda\gamma^\mu\gamma_5\psi\bar{\psi}(x_1)\psi(x_2))|0\rangle - \delta(x-x_2)\langle 0|T(\bar{\psi}(x_1)\gamma^\mu\gamma_5\psi(x))|0\rangle \\ &\quad - \delta(x-x_1)\langle 0|T(\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x_2))|0\rangle = 0.\end{aligned} \tag{4.24}$$

In the course of calculation, anomaly vanishes in the case and divergence of the currents is carefully considered. Namely

$$q_\lambda\Gamma_5^{\lambda\mu}(p_1p_2q) = iq^\mu\Gamma_5 + i\gamma^\mu\gamma_5 S^{-1}(p_2) + i\gamma_5\gamma^\mu S^{-1}(p_1). \tag{4.25}$$

This result suggests that the anomaly associating with WTI may vanish even if the gauge field involves γ_5 coupling.

$$\text{(vii)} \quad \theta_{[\nu]}(x) = \theta_{\mu\nu}(x). \tag{4.26}$$

Using the identity

$$\gamma^\lambda\sigma^{\mu\nu} = \frac{i}{2}\gamma^\lambda\gamma^\mu\gamma^\nu - \frac{i}{2}\gamma^\lambda\gamma^\nu\gamma^\mu \tag{4.27}$$

the corresponding WT identity for tensor current $j^{\lambda\mu\nu} = \bar{\psi}(x)\gamma^\lambda\sigma^{\mu\nu}\psi(x)$ is given by

$$\begin{aligned}&-\delta(x-x_2)\gamma^0\sigma^{\mu\nu +}\gamma^0\langle 0|T(\bar{\psi}(x_1)\psi(x))|0\rangle - \delta(x-x_1)\sigma^{\mu\nu}\langle 0|T(\bar{\psi}(x)\psi(x_2))|0\rangle \\ &\quad - i\partial_\lambda\langle 0|T(\bar{\psi}(x)g^{\lambda\mu}\gamma^\nu\psi(x)\bar{\psi}(x_1)\psi(x_2) - \bar{\psi}(x)g^{\lambda\nu}\gamma^\mu\psi(x)\bar{\psi}(x_1)\psi(x_2) \\ &\quad - \bar{\psi}(x)i\varepsilon^{\lambda\mu\nu\rho}\gamma_\rho\gamma_5\psi(x)\bar{\psi}(x_1)\psi(x_2))|0\rangle = 0.\end{aligned} \tag{4.28}$$

In momentum space, it changes to

$$q^2\Gamma_V^\mu(p_1p_2) = q^\mu q_\nu\Gamma_V^\nu - i\varepsilon^{\lambda\mu\nu\rho}q_\nu q_\lambda\Gamma_{A\rho} + iS^{-1}(p_2)q_\nu\sigma^{\mu\nu} + iS^{-1}(p_1)q_\nu\sigma^{\mu\nu}. \tag{4.29}$$

Obviously, the full vector functions and the full axial–vector functions are coupled with each other. The apparent feature of the identity is that the vertex function Γ_V^μ (fermion's three point function) has the transverse components of itself (this will be discussed in Sect. 5).

$$\text{(viii)} \quad \theta(x) = \theta_{\mu\nu_5}(x). \tag{4.30}$$

Completely analogous to the calculations above, the WTTI for $j_5^{\mu\nu} = \bar{\psi}(x)\sigma^{\mu\nu}\gamma_5\psi(x)$ is derived

$$\begin{aligned}&-\delta(x-x_2)\sigma^{\mu\nu}\gamma_5\langle 0|T(\bar{\psi}(x_1)\psi(x))|0\rangle - \delta(x-x_1)\sigma^{\mu\nu}\gamma_5\langle 0|T(\bar{\psi}(x)\psi(x_2))|0\rangle \\ &\quad - i\partial_\lambda\langle 0|T(\bar{\psi}(x)g^{\lambda\mu}\gamma^\nu\gamma_5\psi(x)\bar{\psi}(x_1)\psi(x_2) - \bar{\psi}(x)g^{\lambda\nu}\gamma^\mu\gamma_5\psi(x)\bar{\psi}(x_1)\psi(x_2)\end{aligned}$$

$$
\begin{aligned}
&- \bar{\psi}(x) i \varepsilon^{\lambda\mu\nu\rho}\gamma_\rho \psi(x)\bar{\psi}(x_1)\psi(x_2))|0\rangle \\
&+ \langle 0|T(J_{[\mu\nu 5]}(x)\bar{\psi}(x_1)\psi(x_2))|0\rangle = 0. \qquad (4.31)
\end{aligned}
$$

The identity for the axial-tensor current is rewritten as

$$
\begin{aligned}
q^2\Gamma_A^\mu(p_1p_2) &= q^\mu q_\nu \Gamma_A^\nu - i\varepsilon^{\lambda\mu\nu\rho} q_\lambda q_\nu \Gamma_{V\rho} + iq_\nu\gamma_5\sigma^{\mu\nu}S^{-1}(p_2) \\
&\quad - iq_\nu\gamma_5\sigma^{\mu\nu}S^{-1}(p_1) + q_\nu J^{[\mu\nu 5]}(q) = 0. \qquad (4.32)
\end{aligned}
$$

Being similar to (4.29), the axial-vertex functions in (4.33) couple with the vector vertex functions. Clearly, a non-trivial Jacobian contributes to the WT identity.

$$
\text{(ix)} \quad \theta_{[\nu]}(x) = \theta_{5\mu\nu\kappa}(x). \qquad (4.33)
$$

As before, the WTI for the high rank tensor current $j^{\lambda\mu\nu\kappa 5} = \psi(x)\varepsilon^{\mu\nu\kappa\rho}\gamma^\lambda\gamma_\rho\gamma_5\psi(x)$ can be given by

$$
\begin{aligned}
&\partial_\lambda\langle 0|T(\bar{\psi}(x)\gamma^\lambda\varepsilon^{\mu\nu\kappa\rho}\gamma_\rho\gamma_5\psi(x)\bar{\psi}(x_1)\psi(x_2))|0\rangle + \delta(x-x_2)\langle 0|T(\bar{\psi}(x_1)\varepsilon^{\mu\nu\kappa\rho}\gamma_\rho\gamma_5\psi(x))|0\rangle \\
&\quad - \delta(x-x_1)\langle 0|T(\bar{\psi}(x)\varepsilon^{\mu\nu\kappa\rho}\gamma_\rho\gamma_5\psi(x_2))|0\rangle = 0. \qquad (4.34)
\end{aligned}
$$

Taking the Fourier transform of the WT identity, it reads off

$$
\begin{aligned}
-q^2\Gamma_T^{\mu\kappa}(p_1p_2) &= q_\lambda q_\rho \varepsilon^{\lambda\mu\kappa\rho}\Gamma_5 + iq^\mu(S^{-1}(p_1)\gamma^\kappa + \gamma^\kappa S^{-1}(p_2) + q^\kappa\Gamma_S + 2m\Gamma_V^\kappa(p_1p_2) \\
&\quad + A(q)) + iq^\kappa(S^{-1}(p_1)\gamma^\mu + \gamma^\mu S^{-1}(p_2) + q^\mu\Gamma_S + 2m\Gamma_V^\mu(p_1p_2) \\
&\quad + A(q)) + \varepsilon^{\kappa\mu\nu\rho}\gamma_\rho\gamma_5 q_\nu S^{-1}(p_2) - \varepsilon^{\kappa\mu\nu\rho}\gamma_\rho\gamma_5 q_\nu S^{-1}(p_1).
\end{aligned}
$$

It is shown that the full tensor vertex function $\Gamma_T^{\mu\kappa}$ is involved the vector vertex Γ_V^μ which come from the longitudinal component equation (4.13).

5 Transverse Component of Variant Green Functions and Full Vertex

As a straightforward application of these WTIs, we can now give the expressions of the full vector axial-vector and tensor vertex functions by solving the above set of coupled equations. In order to illustrate clearly the physical meaning of the identity equation (4.29), the full vector vertex function is decomposed into transverse and longitudinal modes

$$
\Gamma_V^\mu(p_1p_2) = \Gamma_{V(L)}^\mu(p_1p_2) + \Gamma_{V(T)}^\mu(p_1p_2). \qquad (5.1)
$$

Considering the antisymmetry property of $\varepsilon^{\lambda\mu\nu\rho}$ and $\sigma^{\mu\nu}$, we naturally find the identity

$$
q_\mu\Gamma_{V(T)}^\mu(p_1p_2) = 0. \qquad (5.2)
$$

By the help of WT identities (4.12) and (4.18), we can carry out the full vector vertex function from (4.29) and (4.33)

$$
\Gamma_{V(L)}^\mu(p_1p_2) = q^{-2}q^\mu(S^{-1}(p_1) - S^{-1}(p_2)). \qquad (5.3)
$$

This is just the well-known form (4.12).

$$\begin{aligned}\Gamma^{\mu}_{V(T)} &= q^{-2}(q^2 q_\nu \sigma^{\mu\nu} - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} q_\rho \gamma_5 - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} \sigma_{\rho\sigma} q^\sigma \gamma_5) i S^{-1}(p_1) \\ &\quad + q^{-2}(q^2 q_\nu \sigma^{\mu\nu} - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} q_\rho \gamma_5 - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} \sigma_{\rho\sigma} q^\sigma \gamma_5) i S^{-1}(p_2) \\ &\quad - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} q_\rho (2m\Gamma_5(p_1, p_2) + J_{[5]}(q)).\end{aligned}$$

This is quite complicated by contrast with its longitudinal component. The important point is that anomaly has a contribution to the vertex.

By analogy with (5.1), the full axial-vector vertex function is given by

$$\Gamma^{\mu}_{A}(p_1 p_2) = \Gamma^{\mu}_{A(L)}(p_1 p_2) + \Gamma^{\mu}_{A(T)}(p_1 p_2). \tag{5.4}$$

With the longitudinal mode

$$\Gamma^{\mu}_{A(L)}(p_1 p_2) = q^{-2} q^\mu (S^{-1}(p_1)\gamma_5 + S^{-1}(p_2)\gamma_5 + 2im\Gamma_5(p_1 p_2)) + J_{[5]}(q) \tag{5.5}$$

and with the transverse mode

$$\begin{aligned}\Gamma^{\mu}_{A(T)} &= [\varepsilon^{\lambda\mu\nu}_{\kappa} q_\nu q_\lambda (q^2 q_{\nu'} \sigma^{\kappa\nu'} - \varepsilon^{\lambda'\kappa\nu'\rho} q_{\nu'} q_{\lambda'} q^{-4} q_\rho \gamma_5 - \varepsilon^{\lambda'\kappa\nu'\rho} q_{\nu'} q_{\lambda'} q^{-4} \sigma_{\rho\sigma} q^\sigma \gamma_5) \\ &\quad - i\sigma^{\kappa\nu'} q_{\nu'} \gamma_5] q^{-2} S^{-1}(p_1) \\ &\quad + [\varepsilon^{\lambda\mu\nu}_{\kappa} q_\nu q_\lambda (q^2 q_{\nu'} \sigma^{\kappa\nu'} - \varepsilon^{\lambda'\kappa\nu'\rho} q_{\nu'} q_{\lambda'} q^{-4} q_\rho \gamma_5 - \varepsilon^{\lambda'\kappa\nu'\rho} q_{\nu'} q_{\lambda'} q^{-4} \sigma_{\rho\sigma} q^\sigma \gamma_5) \\ &\quad - i\sigma^{\kappa\nu'} q_{\nu'} \gamma_5] q^{-2} S^{-1}(p_2) - \varepsilon^{\lambda\mu\nu}_{\kappa} q_\nu q_\lambda \varepsilon^{\lambda\kappa\nu'\rho} q_{\nu'} q_\lambda q^{-4} q_\rho (2m\Gamma_5(p_1, p_2) \\ &\quad + J_{[5]}(q)) + q^{-2} q^\mu J_{[\mu\nu 5]}(q).\end{aligned} \tag{5.6}$$

It is shown that the full vector and the full axial-vector vertex functions can be expressed entirely and rigorously in terms of two-point functions, the scalar and the pseudo-scalar vertex functions, respectively. In addition, the full tensor vertex function (4.40) can be expressed in detail by substituting the expression of $\Gamma^{\nu}_{V}(x)$ (or $\Gamma^{\nu}_{A}(x)$) into the identity

$$\begin{aligned}\Gamma^{\mu\kappa}_{T(T)} &= -q^{-2} q_\lambda q_\rho \varepsilon^{\lambda\mu\nu\rho} \Gamma_5 \\ &\quad + (iq^\mu \gamma^\kappa + iq^\mu \gamma^\kappa - q_\nu \varepsilon^{\kappa\mu\nu\rho} \gamma_\rho \gamma_5 \\ &\quad + 2mi^2 q^\mu (q^2 q_\nu \sigma^{\kappa\nu} - \varepsilon^{\lambda\kappa\nu\rho} q_\nu q_\lambda q^{-4} q_\rho \gamma_5 - \varepsilon^{\lambda\kappa\nu\rho} q_\nu q_\lambda q^{-4} \sigma_{\rho\sigma} q^\sigma \gamma_5) \\ &\quad + 2mi^2 q^\kappa (q^2 q_\nu \sigma^{\mu\nu} - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} q_\rho \gamma_5 - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} \sigma_{\rho\sigma} q^\sigma \gamma_5)) q^{-2} S^{-1}(p_1) \\ &\quad + (iq^\mu \gamma^\kappa + iq^\mu \gamma^\kappa + q_\nu \varepsilon^{\kappa\mu\nu\rho} \gamma_\rho \gamma_5 + 2mi^2 q^\mu (q^2 q_\nu \sigma^{\kappa\nu} - \varepsilon^{\lambda\kappa\nu\rho} q_\nu q_\lambda q^{-4} q_\rho \gamma_5 \\ &\quad - \varepsilon^{\lambda\kappa\nu\rho} q_\nu q_\lambda q^{-4} \sigma_{\rho\sigma} q^\sigma \gamma_5) + 2mi^2 q^\kappa (q^2 q_\nu \sigma^{\mu\nu} - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} q_\rho \gamma_5 \\ &\quad - \varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} \sigma_{\rho\sigma} q^\sigma \gamma_5)) q^{-2} S^{-1}(p_2) \\ &\quad - iq^{-2} q^\kappa q^\mu \Gamma_5 + 2miq^\kappa q^{-2} (\varepsilon^{\lambda\mu\nu\rho} q_\nu q_\lambda q^{-4} q_\rho (2m\Gamma_5(p_1, p_2) + J_{[5]}(q))) \\ &\quad + iq^{-2} q^\kappa A^\mu(q) - 2miq^\mu q^{-2} (\varepsilon^{\lambda\kappa\nu\rho} q_\nu q_\lambda q^{-4} q_\rho (2m\Gamma_5(p_1, p_2) + J_{[5]}(q))).\end{aligned}$$

The vertex function is very complicated. It is also shown that the contribution of anomaly is indispensable for full vertex.

6 Concluding Remarks

In this paper, we carried out successfully the presentation of the transverse part and longitudinal part of various full vertex functions by solving the complete set of WT identities in a quantized gauge theory (QED). By introducing a postulate of gauge group parameter, the local infinitesimal symmetry transformation preserving the gauge invariance of generating function itself has been revealed successfully. As (2.10) shown, such a collection of gauge transformations connected with the assumption have been verified to meet the properties of gauge group $U(\theta(x))$. The existence of the new symmetry is proved rigorously.

As already described, path-integral method provide a general regularization procedure handling all the anomaly factors associated with WTIs. The evaluation of anomaly function $\Gamma(x)$ has the technical advantage of providing the anomaly factor from integral measure. In particular, beside the previous results such as the chiral anomaly [18–21, 23], a new nontrivial Jacobian equation (3.23) of higher tensor fermionic currents arises. Also, the analysis of (3.15) and (3.16) shows that the resulting anomaly functions depending on regulator $f^d(-(\frac{\lambda_n}{M})^2)$ is not unique.

In sense of the new symmetry, general class of the WT identities corresponding to variant fermionic currents have been completely derived in the coordinate space as well as in the momentum space. As a consequence, the full vector, the full axial-vector and the full tensor vertex functions exactly and completely deduced from the set of WTIs without any Ansatz. In particular, all the full vertex show a important feature that they all contain anomaly terms. On the analogy of the case of low-energy theorems [14–17], perhaps the existence of the anomaly terms may lead to physical consequences.

It is worth remarking the difference between our presentation of WT identities and the results obtained before [7]. First, in our prescription, an effective path integral method serve as a non-perturbative treatment of the theory; while the operator approach in Ref. [7] is ordinary perturbation manner. Second, In former way, WTIs for variants tensor currents automatically contain anomaly terms coming from integral measure and from divergence of Noether's current, rather than from perturbative diagrammatic analysis such as modification by higher-order correction [25], in the latter the contribution of ABJ anomaly is not considered sufficiently for WT identity. Third, path-integral approach provides reliable and complete expressions of full vertex functions. by contrast, the results from the latter are quite complicated.

We would like to point out the above prescription for QED model is general: namely, it can have a useful generalization to a wide class of gauge field theories. Therefore there is a viable probability to eventually derive Full Green function for the case of effective QCD with Fadeev–Popov ghost field.

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